# Multiple Time-scale analysis of Intensity Rate Equations Reveals Stochastic Resonance in VCSELs

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*We provide analytical evidence of stochastic resonance in polarization switching vertical-cavity surface-emitting lasers (VCSEL). Describing the VCSEL by a two-mode stochastic rate equation model and applying a multiple time-scale analysis, we reduce the dynamical description to a single stochastic differential equation. This equation is used as the starting point of an analytical study of stochastic resonance. Simultaneously, we numerically simulate the original rate equations to validate the use of a multiple time-scale analysis on stochastic equations as an analytical tool.*

## Introduction

As early as 1981, Benzi et al.[1, 2] and Nicolis et al.[3] introduced the concept of stochastic resonance to explain the periodicity of ice ages. This phenomenon is characterized by the presence of noise increasing the effect of a small modulation[4]. Recently, stochastic resonance (SR) has also been observed in Vertical-Cavity Surface-Emitting semiconductor Lasers (VCSELs) [5]. VCSELs operating in the fundamental transverse mode usually emit light in one of two specific orthogonal linear polarizations states [6, 7, 8]. In some devices the emitted polarization changes at a specific (switching) current. Around this switching current there is a small region where spontaneous mode hopping is observed between the two modes[9]. When the current is modulated in this region, SR can be observed[10]. In this contribution we theoretically investigate SR in VCSELs. We first show the presence of SR using a simplified two state model [11]. In an alternative approach , we analyze the residence time distribution [12, 13, 14].

## Stochastic rate equations

We model the polarization behavior of VCSELs with the following reduced rate equations that describe the evolution of the carrier density ($\eta$) and the photon density in the two polarization modes ($p_x$ and $p_y$) [15, 9]

$$\frac{dp_x}{dt} = p_x\left[\eta - \varepsilon_{sx}p_x - \varepsilon_{xy}p_y\right] + \tfrac{1}{2}R_{sp} + \tilde{F}_x \tag{1}$$

$$\frac{dp_y}{dt} = p_y\left[\eta + G(J) - \varepsilon_{sy}p_y - \varepsilon_{yx}p_x\right] + \tfrac{1}{2}R_{sp} + \tilde{F}_y \tag{2}$$

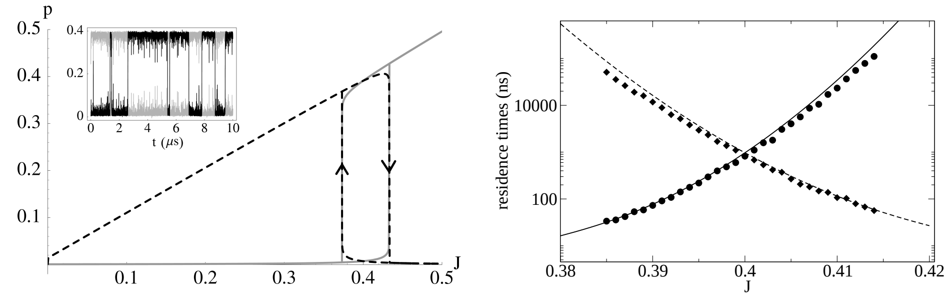
$$\frac{d\eta}{dt} = \frac{J - p_x - p_y}{\rho} - \eta - p_x\left[\eta - \varepsilon_{sx}p_x - \varepsilon_{xy}p_y\right] - p_y\left[\eta - \varepsilon_{sy}p_y - \varepsilon_{yx}p_x\right] + \tilde{F}_n \tag{3}$$

with

$$\langle \widetilde{F}_x(t)\widetilde{F}_x(s)\rangle = 2R_{sp}p_x\delta(t-s) \qquad \langle \widetilde{F}_y(t)\widetilde{F}_y(s)\rangle = 2R_{sp}p_y\delta(t-s) \qquad \langle \widetilde{F}_x(t)\widetilde{F}_y(s)\rangle = 0 \tag{4}$$

The time $t$ is reduced with respect to the carrier life time and $\rho = \frac{\tau_p}{\tau_c} \simeq 10^{-3}$. The parameters $J$, $\varepsilon_{sx,sy,xy,yx}$ are the reduced current and the saturation coefficients. $R_{sp}$ describes the noise strength. The parameter $G(J) = g(1 - \frac{J}{J_s})$ is the current dependent linear dichroism. Using the same approach as in [15, 16], we now deduce a single dynamical equation from Eqs. (1-4) :

$$\dot{p}_y = p_y(J-p_y)\left(\frac{2\delta}{J}p_y - \delta + \frac{G}{J}\right) + \frac{R_{sp}}{2J}(J-2p_y) + \widetilde{F}_y - \frac{\widetilde{F}_x + \widetilde{F}_y}{J}p_y \tag{5}$$




(a) (b)

Figure 1: (a) Numerical solution of Eqs. (1-3) . Realistic parameter values [9], are $\varepsilon_{sx} = \varepsilon_{sy} = 4$, $\varepsilon_{xy} = \varepsilon_{yx} = 8$, $g = 14$, $J_s = 0.4$, $R_{sp} = 0.023$, $\rho = 10^{-3}$. The inset shows the random hopping between the two polarization modes. Black curves correspond to the $p_y$ mode, and the grey curves to the $p_x$ modes. (b) The residence times of the $p_x$ (full line and circles) and $p_y$ (dashed line and diamonds) mode throughout the bistable region. Comparison of Eq. (6)(lines) with numerical simulation of Eqs. (1-3) (circles and diamonds).

Equation (5) is the result of the MTSA approach. Figure 1 (a) shows the stable steady state solutions. We call the mode which starts lasing at threshold the $p_y$ mode, which implies that $g$ is positive. When the current is set at $J = J_s$, spontaneous hopping between the two pure-modes occurs (see inset figure 1 (a)) [9, 16].

SR takes place when we harmonically modulate the current with a period $T$ inside the bistable region, $J = J_s + J_m \sin(\omega t + \phi)$, with $\omega = 2\pi/T$. Depending on the noise strength, the output will follow the modulation with more or less succes. To quantify the level of succes, we will define two indicators, both based on the switching rates.

We can obtain the switching rates between the two polarization modes from the Fokker-Planck equation of the probability density function of $p \equiv p_y$. We will call $p \simeq J$ mode the "+"state and $p \simeq 0$ the "-"state. The mean residence times of these two states are then given by [17, 18, 10]:

$$t_{\pm} = \frac{2J\pi\delta}{J^2\delta^2 - G^2} \operatorname{erf}\left(\frac{G \pm J\delta}{2\sqrt{R_{sp}\delta}}\right) \operatorname{erfi}\left(\frac{G \pm J\delta}{2\sqrt{R_{sp}\delta}}\right) \tag{6}$$

In Fig. 1(b) we compare Eq. (6) with numerical simulations of Eqs. (1-3). We see that the residence times scale over three orders of magnitude with the current. In the whole region the correspondence between the numerical and the analytical approach is quite good.

## Analysis of the SR

### Two state model [11]

As the intra-well relaxation time of $p$ is much shorter than the residence times, we can make a two-state approximation: we model the continuous system as being in either the off-state, or the on-state, filtering all the information except in which potential well the particle resides at time $t$ [11]. We define $n_+(t)$ and $n_-(t)$ as the probability that the system is in the $p_y$ or $p_x$ state at time $t$. These probabilities change according to the following two master equations [11, 4] :

$$\dot{n}_{\pm}(t) = W_{\pm}(t)n_{\mp}(t) - W_{\mp}(t)n_{\pm}(t) \tag{7}$$

In Eq. (7), $W_{\pm}(t)$ are the switching rates to the on/off state, i.e. the inverse of the residence times in Eq. (6). If we modulate the current, these rates will indeed depend sinusoidally on the time. The average of the intensity of one of the polarization modes is approximately, (assuming that $p = 0$ in the off-state and $p = J$ in the on state)

$$\langle p \rangle = p_{\text{off}} n_- + p_{\text{on}} n_+ \simeq J_s n_+ \tag{8}$$

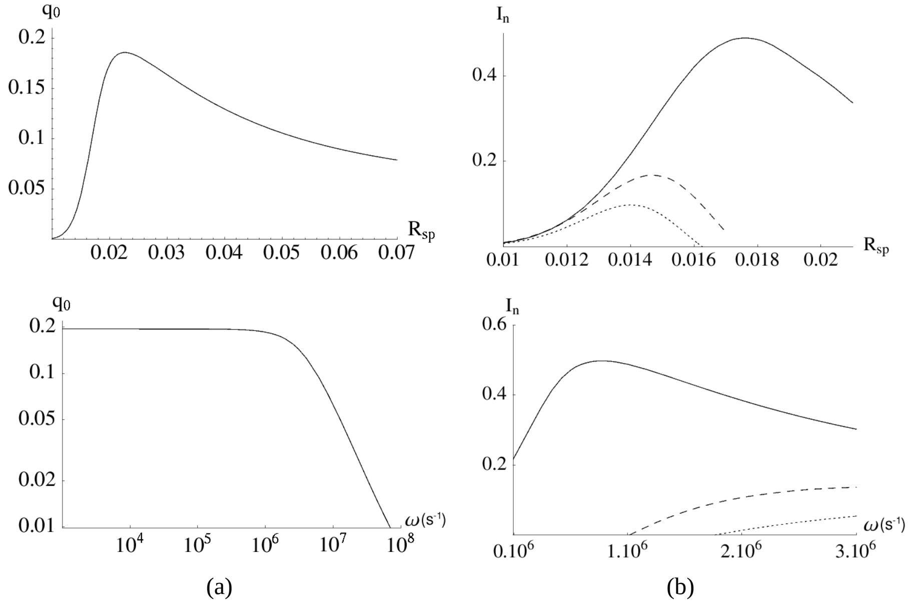



Figure 2: (a) Parameter $q_0$ as a function of $R_{sp}$ (top), and ω (bottom). Parameter values are $\delta = 4$, $g = 14$, $J = J_s + J_m \sin(\omega t)$, $J_s = 0.4$, $J_m = 0.005$, $\omega = 10^6 s^{-1}$ in the top graph and $R_{sp} = 0.023$ in the bottom graph. Perfect synchronization would give a value of $q_0 = \frac{2J_s}{\pi}$ (b) Indicators $I_n$ (see Eq. (10)) of SR. The top figure shows the indicator as a function of $R_{sp}$ for a constant pulsation ($\omega = 10^6 s^{-1}$), the bottom figure shows the indicator as a function of the pulsation for a constant noise strength ($R_{sp} = 0.0175$). The black, dashed and dotted curve correspond to $I_1$, $I_2$ and $I_3$.

We define the indicator $q_0$ as the first Fourier coefficient of Eq. (8). In Fig. 2 (a), $q_0$ is plotted as a function of the noise strength for a constant frequency, and as a function of the frequency for a constant noise strength, showing the typical SR[10].

**Residence Time Distribution**

Another way to investigate SR is to look at the statistical properties of the residence times [12, 19]. When the potentials are constant, the residence times have the usual exponential distribution. If the current is modulated sinusoidally as in Eq. (**??**), this distribution has the form:

$$P_{m_\pm}(t) = t_\pm(J)^{-1} \exp(-\int_0^t \frac{ds}{t_\pm(J)}) \tag{9}$$

The first peak in the distribution function represent switches which are synchronized with the modulation signal, whereas the following peaks represent events where the system did not switch for one or more periods. We now introduce an indicator calculating the area under the peaks at $T/2$, $3T/2$, etc. after subtraction of the background :

$$I_n = \int_{nT-\frac{3}{4}T}^{nT-\frac{1}{4}T} \left(P_m(J = J_0 + J_m \sin[\omega t]) - P_m(J = J_0)\right) dt \tag{10}$$

We have a SR if the indicator $I_1$ attains a maximum, while the others (i.e. $I_2$, $I_3$, $\ldots$) do not. In Fig. 2 (b) we plot $I_1$, $I_2$ and $I_3$ as a function of the frequency(for constant noise strength) and as a function of the noise strength (for constant frequency). We see that SR appears both as a function of the noise strength and as a function of the frequency (also called bona fide resonance [19, 10]).

## Conclusion

In this paper we have theoretically studied SR between two stable polarization states in VCSELs. We have analyzed the resonance in two ways. We first used a two state model which only considers in which potential minimum (corresponding to either the $p_x$ or the $p_y$ polarization state) the system is. Alternatively, we have proposed an indicator, similar to ones previously proposed by [4, 13, 20, 21], based on the residence time distribution. The indicator shows a resonant behavior both as a function of the noise strength and as function of the frequency, so called *bona fide* resonance.

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