Sommerfeld Precursor: From a Homogeneous Dielectric Medium to a Photonic Crystal

R.Uitham and B.J. Hoenders

Materials Science Center, Rijksuniversiteit Groningen,
Nijenborgh 4, 9747 AG Groningen, The Netherlands

We calculate the transverse electric polarization transmission coefficient for a finite one dimensional photonic crystal with the use of the discrete Mellin transform. The Sommerfeld precursor for an electromagnetic pulse that propagates through a homogeneous, dispersive dielectric medium with absorption is derived. We indicate where the modification to this precursor arises when the homogeneous medium is replaced with the photonic crystal. The consistency of the short-wavelength precursor with macroscopic Maxwell theory is discussed.

Introduction

Photonic crystals (PC's) of dimension n are dielectric materials with periodic refractive index in n orthogonal directions. In PC's, electromagnetic (EM) waves of certain frequencies can not propagate because of multiple reflections. This property turns the three-dimensional PC into a promising candidate material for information technology, since it can be used as a small-scale and low-loss waveguide for EM signals. The final goal of the research is to calculate the effect of a PC on a generic EM pulse that propagates through it. Here the 'beginning of the pulse' is investigated: the part immediately behind its wavefront. For homogeneous materials this region is well known [2]. It is the domain of the Sommerfeld precursor.

Transmission Coefficient for a Photonic Crystal

The pulse propagates through a one dimensional PC that consists of N identical layers (unit cells) of two dielectric media with alternating refractive indices $n_i(\omega),\ i = 1,2$. It is modelled similarly as in [3],

$$n(x;\omega) = \begin{cases} n_1(\omega) & x < 0, \quad x > N\Lambda \\ n_2(\omega) & 0 < x < b \\ n_1(\omega) & b < x < \Lambda \\ n(x+\Lambda;\omega) & 0 < x < (N-1)\Lambda \end{cases} \tag{1}$$

with $\Lambda = a + b$. The complex refractive indices

$$n_i(\omega) = 1 + \frac{\omega_{p,i}^2}{\omega_i^2 - \omega^2 - i\gamma_i\omega} \tag{2}$$

contain dispersion and absorption. $\omega_{p,i}$ is the plasma frequency of medium i, ω_i its single atomic resonant frequency and γ_i models the effects of impurities and coupling of the atomic energy states to the photon states of the field, which is treated classically here. The x-axis is chosen in the direction of periodicity of the PC. The propagation axis of the pulse and the x-axis span the plane of incidence, the xz-plane. The electric component of a transversely electric (TE) polarized EM field is $\mathbf{E} = E_y = E$. Its initial momentum is given just left from the PC at $\mathbf{x} = \lim_{\varepsilon\to 0}(-\varepsilon, 0, 0)$ as $(k_x, k_z) = (k_{1x}, \beta)$. The momentum in medium i satisfies $k_{ix}^2 + \beta^2 = \frac{\omega^2}{c^2} n_i^2$. The E-field is then given in the structure of (1) as [3]

$$E(\mathbf{x}, t) = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}\omega \; e^{-i\omega t + i\beta z} E(x;\omega) \tag{3}$$

with, in the n-th unit cell, which extends from $x = (n-1)\Lambda$ to $x = n\Lambda$, $n = 1, 2, \ldots, N$,

$$E(x;\omega) = \begin{cases} E_{1,n} e^{ik_{1x}(x-n\Lambda)} + E'_{1,n} e^{-ik_{1x}(x-n\Lambda)} & n\Lambda - a < x < n\Lambda \\ E_{2,n} e^{ik_{2x}(x-n\Lambda+a)} + E'_{2,n} e^{-ik_{2x}(x-n\Lambda+a)} & (n-1)\Lambda < x < n\Lambda - a \end{cases} \tag{4}$$

The field at $x < 0$ is denoted as $E_{1,0}$ and the field at $x > N\Lambda$ as E_{N+1}. Continuity of E and H_z at each interface implies

$$\begin{pmatrix} E_{1,n} \\ E'_{1,n} \end{pmatrix} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}^n \begin{pmatrix} E_{1,0} \\ E'_{1,0} \end{pmatrix} \tag{5}$$

with (for TM-polarization the entries are slightly different)

$$A = e^{-ik_{1x}a}\left[\cos k_{2x}b - \frac{i}{2}\left(\frac{k_{2x}}{k_{1x}} + \frac{k_{1x}}{k_{2x}}\right)\sin k_{2x}b\right]; \; B = e^{ik_{1x}a}\left[\frac{-i}{2}\left(\frac{k_{2x}}{k_{1x}} - \frac{k_{1x}}{k_{2x}}\right)\sin k_{2x}b\right] \tag{6}$$

and $C = B(i \to -i)$ and $D = A(i \to -i)$. Instead of following the usual procedure to use a matrix identity to obtain the reflection and transmission coefficients in terms of Chebyshev polynomials, we use the discrete Mellin transform to obtain more transparent relations [1],

$$E_{1,n} = \left(\frac{\alpha^{-n-1} - \alpha^{n+1} - A(\alpha^{-n} - \alpha^n)}{\alpha^{-1} - \alpha}\right) E_{1,0} + B\frac{\alpha^{-n} - \alpha^n}{\alpha - \alpha^{-1}} E'_{1,0} \tag{7}$$

$$E'_{1,n} = \left(\frac{\alpha^{-n-1} - \alpha^{n+1} - D(\alpha^{-n} - \alpha^n)}{\alpha^{-1} - \alpha}\right) E'_{1,0} + C\frac{\alpha^{-n} - \alpha^n}{\alpha - \alpha^{-1}} E_{1,0} \tag{8}$$

with $\alpha = \frac{1}{2}(A+D) + \sqrt{\frac{1}{4}(A+D)^2 - 1}$. The reflection and transmission coefficients follow as

$$r_N = \left(\frac{E'_{1,0}}{E_{1,0}}\right)_{E'_{1,N}=0} = \frac{C(\alpha^N - \alpha^{-N})}{\alpha^{N+1} - \alpha^{-(N+1)} - D(\alpha^N - \alpha^{-N})} \tag{9}$$

$$t_N = \left(\frac{E_{1,N}}{E_{1,0}}\right)_{E'_{1,N}=0} = \frac{\alpha^{N+1} - \alpha^{-(N+1)} - (A + Br_N)(\alpha^N - \alpha^{-N})}{\alpha - \alpha^{-1}} \tag{10}$$

The Incoming Pulse

Let $E(t)$ denote the amplitude of the TE electric field of the incident pulse at $\mathbf{x} = \lim_{\varepsilon\to 0}(-\varepsilon, 0, 0)$, where its propagation direction is specified as above. Let this amplitude be a continuous function of time, possibly nonzero but finite for a finite amount of time $T > 0$, such that

$$\begin{array}{ll} E(t) \in \mathbb{R} & t \in (0,T) \\ E(t) = 0 & t \notin (0,T) \end{array} \tag{11}$$

With $E(t)$ a continuous function of time, the field for $x < 0$ can be expressed as

$$E_{1,0}(\mathbf{x},t) = \frac{1}{\sqrt{2\pi}}\int_S d\omega\, E_{1,0}(\omega)e^{-i\omega t+ik_{1x}x+i\beta z} \tag{12}$$

$$E_{1,0}(\omega) = \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{+\infty}\omega_n E(\omega_n)\frac{1-(-1)^n e^{i\omega T}}{\omega^2-\omega_n^2};\, E(\omega_n) = \frac{2}{T}\int_0^T dt\, E(t)\sin\omega_n t \tag{13}$$

and with S an integration path traversed from $\omega = +\infty$ to $\omega = -\infty$ through the upper half complex ω-plane slightly above the real axis. The transmitted field at $x \geq N\Lambda$ is

$$E_{1,N}(\mathbf{x},t) = \frac{1}{\sqrt{2\pi}}\int_S d\omega\; t_N(\omega)E_{1,0}(\omega)\exp\Big(-i\omega t + ik_{1x}(x-N\Lambda)+i\beta z\Big) \tag{14}$$

This signal is investigated with the use of integration path deformations. Therefore it is important to locate all its singularities in the complex ω-plane. They are:

- first-order poles from the Fourier representation (13) of $E_{1,0}$ at $\omega = \pm\omega_n$,
- branch cuts from the dispersion relations $k_i(\omega) = c^{-1}\omega n_i(\omega)$ along $\mathrm{Im}(\omega) = -i\gamma_1/2$, see (15) and (16),
- poles of t_N in the lower half ω-plane (on behalf of causality).

The Sommerfeld Precursor

The equation $\tau = 0$ for the parameter $\tau = t - c^{-1}(x\cos\theta_1 + z\sin\theta_1)$ gives the wave front of the pulse. The situation $\tau < 0$ corresponds to a position $\mathbf{x}$ at a time τ ahead of the wavefront and $\tau > 0$ corresponds to a position at a time τ behind the wavefront. We are interested in the situation just behind the wave front, i.e. for τ a little larger than zero, but smaller than T. For $\tau < T$, the second term in (13) vanishes. With the auxiliary variables

$$\kappa_{1x} = k_{1x} - \frac{\omega\cos\theta_1}{c} = \frac{\omega\cos\theta_1}{c}\left[\sqrt{1+\frac{\omega_p^2}{\omega_1^2-\omega^2-i\gamma\omega}}-1\right] \tag{15}$$

$$\kappa_{1z} = \beta - \frac{\omega\sin\theta_1}{c} = \frac{\omega\sin\theta_1}{c}\left[\sqrt{1+\frac{\omega_p^2}{\omega_1^2-\omega^2-i\gamma\omega}}-1\right] \tag{16}$$

the transmitted signal (14) for $0 < \tau < T$ can be written as a function of τ,

$$E_{1,N}(\mathbf{x},t) = \frac{1}{2\pi}\int_S d\omega\; t_N(\omega)e^{-ik_{1x}N\Lambda}\sum_{n=1}^{+\infty}\omega_n E(\omega_n)\frac{e^{-i\omega\tau+i\kappa_{1x}x+i\kappa_{1z}z}}{\omega^2-\omega_n^2} \tag{17}$$

The integration path S may be deformed to a semicircle with very large radius in the upper half ω-plane to give the same result since there are no singularities above S. Then, we may attach to this path another semicircle in the lower half plane since the integral vanishes for $|\omega| \to \infty$ in the lower half plane. The combined path is denoted as C. Now we impose a cutoff on the infinite sum in (13): the incident pulse is approximated with a finite number L of sinuses, such that $|\omega| >> \omega_L$ for ω on C. Then (17) can be approximated as

$$E_{1,N}(\mathbf{x},t) = \frac{1}{2\pi}\oint_C d\omega\; t_N(\omega)e^{-ik_{1x}N\Lambda}\sum_{n=1}^{L}\omega_n E(\omega_n)\frac{e^{-i\omega\tau+i\kappa_{1x}x+i\kappa_{1z}z}}{\omega^2} \tag{18}$$

Expand the square roots in (15) and (16) which are of the form $\sqrt{1+\varepsilon}$ up to first order in ε and use $|\omega|^2 >> \omega_1^2, |\gamma\omega|$ to find $\kappa_{1x}x = -\cos\theta_1 \frac{\omega_p^2}{2c}\frac{x}{\omega} = -\frac{\xi}{\omega}$ and $\kappa_{1z}z = -\sin\theta_1 \frac{\omega_p^2}{2c}\frac{x}{\omega} = -\frac{\eta}{\omega}$. The factor $t_N(\omega)e^{-ik_{1x}N\Lambda}$ in (18) should as well be expanded around $\frac{1}{\omega} = 0$: its zeroth order term equals one and the first order term gives the effect of the N layers on the propagation. This last term must still be calculated and is not taken into account at this stage. The field (18) in a homogeneous medium is thus (with $\zeta = \xi + \eta$)

$$E_{1,N}(\mathbf{x},t) = \frac{1}{2\pi}\oint_C \mathrm{d}\omega \sum_{n=1}^{L} \omega_n E(\omega_n) \frac{e^{-i\omega\tau - i\zeta/\omega}}{\omega^2} = \sqrt{\frac{\tau}{\zeta}} \sum_{n=1}^{L} \omega_n E(\omega_n) J_1(2\sqrt{\tau\zeta}) \qquad (19)$$

with J_1 the first order Bessel function. This result is similar as for the case of a sine wave terminated on one side that propagates through a non-absorptive medium as studied in [2]. The initial period τ_0 of the transmitted signal equals the initial period of $J_1(2\sqrt{\tau\zeta})$, which is $\sim 10^{-22}$s. The initial amplitude is extremely small: a spectral component $E(\omega_n)$ of the incident wave is multiplied with a factor $(\tau_0)^{1/2}(\zeta)^{-1/2}\omega_n \sim 10^{-23}\omega_n x^{-1}$ after x meters of propagation. The initial period corresponds to an initial wavelength of $\sim 10^{-14}$m.

Discussion

The frequency dependence of the refractive indices has been derived with a quantum mechanical atomic theory and makes sense even in the limit $\omega \to \infty$. A Fourier representation of EM fields contains all frequency components up to infinity, corresponding to arbitrary small wavelengths. Such a representation is merely a mathematical decomposition and may always be used to represent a physical pulse. The model of refractive indices as piecewise constant functions of position, however, poses a limit to the applicability of the theory. Only the propagation of EM fields with wavelengths much larger than the scale of atomic distances is described properly hence for the result (19) the theory has been extrapolated out of its valid regime. The Sommerfeld precursor of initial wavelength $\sim 10^{-14}$m has not yet been measured experimentally, but there are recent developments in theory [5].

References

[1] B.J. Hoenders and M. Bertolotti, ”Coherence Theory of Electromagnetic Wave Propagation through Stratified N-Layer Media”, submitted for publication

[2] L. Brillouin, *Wave Propagation and Group Velocity*, New York and London: Academic Press, 1960

[3] P. Yeh, Optical Waves in Layered Media, New York: John Wiley and Sons, 1988

[4] J.D. Jackson, Classical Electrodynamics, New York: John Wiley and Sons Inc., 1998, ch. 6, p. 249.

[5] K.E. Oughstun an Hong Xiao, ”Failure of the Quasimonochromatic Approximation for Ultrashort Pulse Propagation in a Dispersive, Attenuative Medium”, *Physics Review Letters*, vol. 78, no. 4, 1997.