Exact Theory for Scattering of Waves by Objects with Non-Standard Shapes.

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The exact theory for scattering of waves by objects of which the boundaries are only partially part of a separable coordinate system, like e.g. a N-layer structure with horizontally oriented layers, is developed. It is shown that it is possible to describe the scattering process still in exactly the same way as in the case of the exactly solvable classical scattering problems: A single mode outside the object couples only to one single mode inside the medium.

Exact Scattering and non-standard Boundaries

The analysis of scattering- and boundary value problems is a very important part of physics and has been extensively explored since the 18^{th} century [1–4]. One of the key features for the possibility to obtain analytical solutions in *closed* form for these problems is that the pertinent scalar- or vectorial wave equations are linear and admit a potential or refractive index for which the equations are still separable. Moreover, another requirement, *essential* for the solution of scattering- and boundary value problems in closed form is that the geometry of the scatterer "fits" with the geometry of the separable potential or refractive index. By this we mean that the level surfaces of the potential or refractive index must coincide fully with the boundary surface(s) of the scatterer. Then the field can be calculated employing the technique of eigenfunction expansions. These eigenfunctions are generated by the set of ordinary differential equations resulting from the separation of the original partial differential equation(s), viz. the scalar- or vectorial wave equation [1], [2].
Unfortunately, no simple theory exists for the calculation of fields generated by the scattering of incoming wavefields by media with so-called *nonfitting boundaries*. By this we mean the following: The "simple" exactly solvable boundary value problems in mathematical physics all share one property: the boundaries of the various geometries involved coincide *fully* with the coordinate surfaces of the various separable coordinate systems for the (vector) wave equation. We mention e.g. scattering of an incoming wave by a half plane, a complete sphere, ellipsoid or cylinder filled with a homogeneous- or a layered medium. The boundaries of all these objects coincide *fully* with a surface

of a separable coordinate system. But, if we consider e.g. scattering by a wedge or a half sphere the theory becomes much more difficult and no theory has been developed yet in case these objects consist of materials with e.g. a layered structure or a radially dependent refractive index. Also if we consider scattering by the objects shown in Fig.1 no exact theory exists in the sense we defined above, viz. the solution of such problems is not known in terms of a series of eigenfunctions with known coefficients.

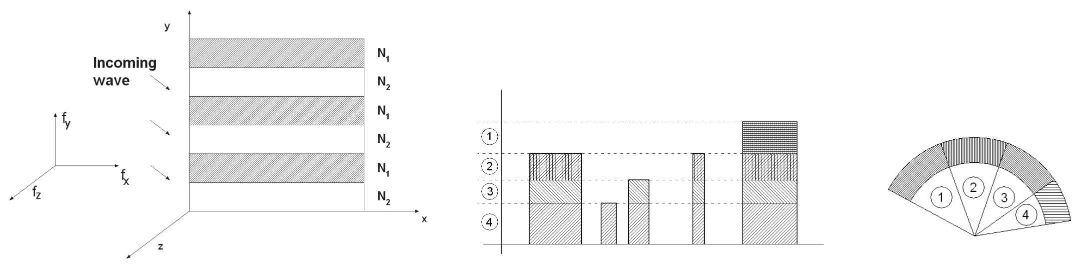

Figure 1: Scattering by a horizontal set of N-layers, a telegrapher's wave- and an insect's eye type of surface. The shaded parts can have different values for the refractive index.

The physical explanation for a yet non-existent simple theory for this case stems from the following observation: A single mode of the field outside the medium no longer couples to *one* single mode inside the medium but to an, in general, *infinite* number of modes. This observation is corroborated analysing the results of the Wiener-Hopf technique [5] . This technique is e.g. used for the description of scattering of waves by a half-plane- or a half cylinder and connects an infinity of modes above (outside) and below (inside) the plane (cylinder) with one mode outside the object, viz. e.g. a plane wave. As an example supporting this statement one could think of a slab built up by N-layers with different constant indexes of refraction such that the layers are perpendicular to the planes of the slab. Then a plane wave mode outside the slab couples to an infinity of modes for the layered medium, whereas the plane wave mode couples to only *one* inside mode if the layers are oriented parallel to the boundary planes of the slab. We also would like to mention the application of this reasoning for the case of a layer covered by rectangular "multiscale" protrusions, (telegrapher's wave surface): Each one of the free space modes of the incoming wave couples to an infinite set of the pertinent modes of the layer containing the rectangular protrusions. ("Multiscale" protrusions are defined as protrusions with different lengths, widths and heights.)

It is shown in our presentation however that our new type of modes, (which are **over-complete!**), have the remarkable property that still one "outside the object" mode couples to one "inside the object" mode. *It is exactly this particular property of one- to one coupling of our new set of modes that enables us to solve the pertinent scattering-boundary value problems for non-fitting boundaries in exactly the same way as done in case of the well-known conventional exactly solvable scattering- boundary value prob-*

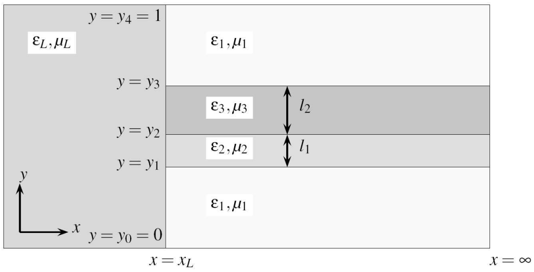

Figure 2: Geometry used for the numerical calculation and the field distribution .

lems! Using these new mode expansions the scattering of an electromagnetic wave by the scattering geometries drawn in Fig.1 will be analysed.
The mentioned basic property of our theory mentioned above, i.e the property that each "outside the object" mode, say ψ_m, couples to one "inside the object" mode, say ψ_M is mathematically formulated in terms of the following completeness relation:

$$\delta\left(y-y'\right) = \sum_n \frac{\psi_m\left(f_n^2;y\right)\psi_M\left(f_n^2;y'\right)}{N'\left(f_n^2\right)}, \tag{1}$$

where the sum is over numbers f_n^2 which are the roots of a known function $N\left(f_n^2\right)$. This relation shows that a field distribution, say $\eta(y)$, can be expanded in either function $\psi_M\left(f_n^2;y'\right)$ or $\psi_m\left(f_n^2;y\right)$ with coupling coefficients $\int \eta(y')\psi_{(m,M)}\left(f_n^2;y\right)dy'$, *depending only on the conjugate mode!!*
For further details we refer to [6]. In the next section we will calculate the fields that are scattered from an interface between a homogeneous medium and a medium with one layer with two slabs that have finite widths. The geometry has been depicted in Fig.2.

Numerical results

Slab parameter values			
	$l_2 = 200$nm	$l_3 = 300$nm	
$\omega_{p1} = 1.5$ $\omega_1 = 4.0$ $\gamma_1 = 0.10$	$\omega_{p2} = 4.5$ $\omega_2 = 4.0$ $\gamma_2 = 0.20$	$\omega_{p3} = 2.0$ $\omega_3 = 2.5$ $\gamma_3 = 0.15$	$\omega_{p4} = 1.5$ $\omega_4 = 4.0$ $\gamma_4 = 0.10$

Table 1: Slab parameter values, where frequencies are in units of 10^{16}rad/s. All media have $\mu_j = \mu_0$

The analytical results obtained above will now be numerically evaluated for the following geometry. At $x < 0$, there is a vacuum. At $x > 0$, we have a dielectric medium that consists of two horizontal layers in between another dielectric medium. The dielectric

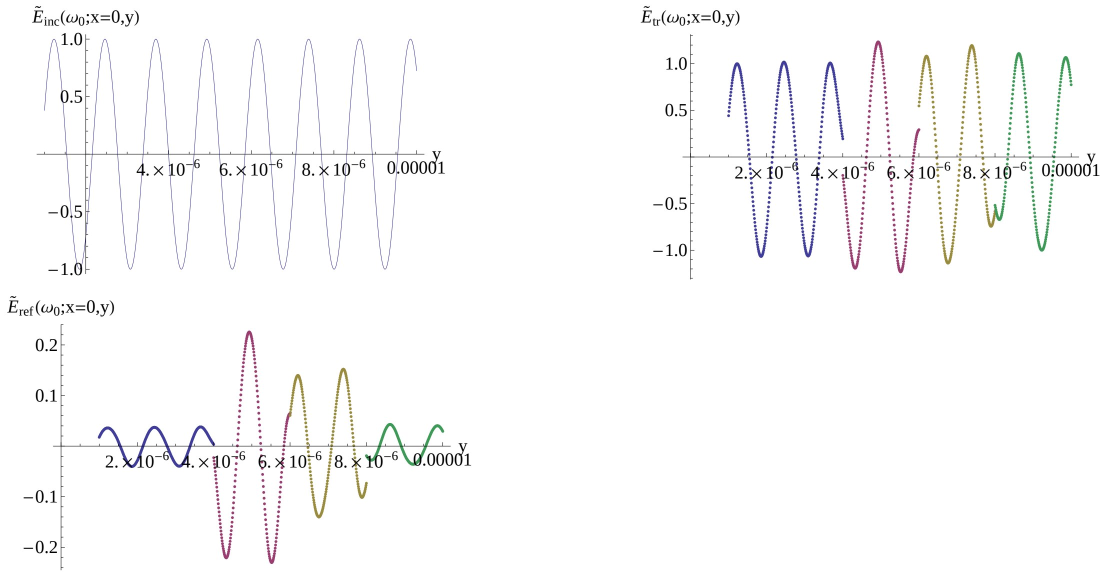

Figure 3: Distributions of the incoming, transmitted and reflected fields at the interface at x = 0 between a homogeneous medium to the left and a horizontal set of two layers to the right.

functions $\varepsilon_j(\omega)$, $j = 1\cdots4$ for the various layers, read as

$$\varepsilon_j(\omega) = \varepsilon_0 + \frac{\varepsilon_0\omega_{pj}^2}{\omega_j^2 - \omega^2 - 2i\gamma_j\omega} \qquad j = 1,\ldots,4. \tag{2}$$

The numerical values of the parameters in Eq. (2) are given in Table 1. Note that the first and fourth layer are given identical optical properties.
The medium is illuminated by the monochromatic incoming field specified above, where we take for the angular frequency $\omega_0 = 4.0\cdot10^{15}$rad/s and for the angle $\theta = \frac{\pi}{8}$. The field distributions at $x = 0$ of this incoming field and the resulting transmitted and reflected fields that were calculated with the above formulae have been plotted in Fig.3.

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