Photon wave function description of space-time entanglement in quantum imaging

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Few photon systems are best described by their wave function rather than by the usual quantum field formalism. Until recently, the existence of a wave function for photons has been an object of controversy. Important breakthrough happened during the last ten years, sharpening the wave function concept and giving it a precise definition. We show how the photon wave function proposed by Bialynicki-Birula and Sipe connects to experimentally measurable quantities such as multi-photon quantum correlation functions and use it to investigate space-time entanglement of photon pairs produced in parametric down-conversion.

Introduction

Few photon systems are best described by their wave function rather than by the usual quantum field formalism. Until recently, the existence of a wave function for photons has been an object of controversy because many people believed that one cannot assign a position-representation wave function to a photon because of the lack of a proper photon-position operator $\hat{\mathbf{r}}$. However, it has been shown that a photon-position operator *can* be constructed [1, 2]. Its Cartesian components are commuting Hermitian operators and satisfy canonical commutation relations $[\hat{r}_k, \hat{p}_l] = i\hbar\delta_{kl}$ with $\hat{\mathbf{p}}$, the photon momentum operator. The eigenfunctions of $\hat{\mathbf{r}}$ are transverse waves that can be interpreted as localised-photon states [3]. The localised photon states are not ideal Dirac functions like in non-relativistic quantum mechanics. Nevertheless, they form a basis that permits to define the *position* representation wave-function of single photons as linear combinations of these localised wave-packets. A position representation for N-photon systems can be defined similarly by taking the tensor product of single-particle Hilbert spaces.

In this paper, we elucidate the connection between the wave function of a N-photon system, which is a mathematical representation of the system state, and experimentally measurable quantities such as multi-photon quantum correlation functions. We then show how the use of the wave-function formalism (WFF) can enlighten our physical understanding of quantum optical systems by calculating how the entanglement of a photon pair produced by parametric down-conversion propagates in space and time.

Photon wave function and optical correlation functions

There is an arbitrariness in the definition of the photon position operator [1]. Therefore, several equally valid wave functions can be found in the literature. Here, we use the wave function $\bar{\boldsymbol{\psi}}(\mathbf{r}, t) = [\boldsymbol{\psi}_+(\mathbf{r}, t), \boldsymbol{\psi}_-(\mathbf{r}, t)]^T$ advocated by Bialynicki-Birula [4, 5] because it is directly related to the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ associated to a single photon: $\boldsymbol{\psi}_\pm = \left(\epsilon_0^{1/2}\mathbf{E}^{(+)} \pm \mathrm{i}\mu_0^{-1/2}\mathbf{B}^{(+)}\right)/\sqrt{2}$, where $X^{(+)}$ means the positive frequency part of field

X. The $\bar{\boldsymbol{\psi}}$-object is 6-component field (a "bi-vector") made of two ordinary vector fields $\boldsymbol{\psi}_+$ and $\boldsymbol{\psi}_-$ representing a circularly polarised photon with positive and negative helicity, respectively. This representation is analogue to Dirac's bi-spinor representation of a relativistic electron. In free-space, each vector component has a Fourier expansion that reads

$$\boldsymbol{\psi}_\pm(\mathbf{r},t) = \int \mathrm{d}^3k\,\sqrt{\hbar k c}\,\mathbf{e}_\pm(\mathbf{k})\;f_\pm(\mathbf{k})\,\frac{\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-kc\ t)}}{(2\pi)^{3/2}}, \tag{1}$$

where $k = |\mathbf{k}|$ and $\mathbf{e}_\pm(\mathbf{k})$ are the unit circular polarisation vectors for photons propagating in the $\mathbf{k}$-direction. Normalisation is such that the complex coefficients $f_\pm$ satisfy $\sum_{h=\pm}\int \mathrm{d}^3k\,|f_h(\mathbf{k})|^2 = 1$. The wave-function $\bar{\boldsymbol{\psi}}$ transforms as an elementary object under Lorentz transformation. Multi-photon wave functions are obtained from tensor products of single-photon ones.

The value of the wave function of a non relativistic particle at some space-time point $(\mathbf{r},t)$ is directly related, through the modulus-square operation, to the probability of finding the particle at that point. We expect a similar property for photon wave functions. In a N-photon quantum optical system, the joint probability to detect σ_i-polarised photons at space-time points $q_i = (\mathbf{r}_i, t_i)$ $(i \in \{1,...,N\})$ is proportional to the correlation function

$$C^{(N)}_{\sigma_1...\sigma_N}(q_1,...,q_N) = \langle \hat{E}^{(-)}_{\sigma_1}(q_1)\ldots\hat{E}^{(-)}_{\sigma_N}(q_N)\hat{E}^{(+)}_{\sigma_N}(q_N)\ldots\hat{E}^{(+)}_{\sigma_1}(q_1)\rangle. \tag{2}$$

In a recent paper [6], Smith and Raymer analyse the connection between the Bialynicki-Birula wave function and correlation functions for $N = 1$ and $N = 2$. Not surprisingly, they find out that the connection is not straightforward because the square on the wave function contains terms that are dependent on the magnetic field. They argue that the discrepancy comes from the fact that (2) only measures joint localisation when the detectors are sensitive to the electric field, the square of the wave function being a more general measure of joint localisation. If we had detectors sensitive to the magnetic field (for instance, a magnetic dipole transition) Eq. (2) would be of no relevance while the square of the wave function would still contain the right information about detection probability. In other words, when calculating joint detection probabilities by squaring the wave function, we must ignore magnetic terms to get the right predictions for electric-type detectors and ignore electric terms for magnetic-type detectors. The problem with this point of view is that, for any practical application of the WFF, there is no way to know which terms of the wave function have a magnetic/electric origin (except for tracing them back through calculation).

To circumvent this fundamental difficulty, we modify the definition of the wave function in such a way that it still contains all the information about both electric and magnetic fields but has a simple connection to the correlation functions (2) that are relevant for photoelectric detection:

$$\boldsymbol{\Psi}(\mathbf{r},t) = \boldsymbol{\psi}_+(\mathbf{r},t) + \boldsymbol{\psi}_-(\mathbf{r},t) = \sqrt{2\epsilon_0}\,\mathbf{E}^{(+)}(\mathbf{r},t). \tag{3}$$

This representation of the photon field was first introduced by Sipe [7]. Interestingly, since $\boldsymbol{\psi}_+$ and $\boldsymbol{\psi}_-$ are orthogonally polarised, they never mix: if $\boldsymbol{\Psi}$ is given, $\boldsymbol{\psi}_+$ and $\boldsymbol{\psi}_-$ can be deduced. Therefore the information content in the vector function $\boldsymbol{\Psi}$ is the same as in the bi-vector field $\bar{\boldsymbol{\psi}}$: both representations are equivalent. Sipe's single-photon wave function is proportional to the positive frequency part of the E-field. In optics, this complex field appears very often in the context of linear and non linear propagation of light and is called the *analytical* E-field. For a N-photon state, the wave function is a tensor whose elements $\Psi_{\sigma_1...\sigma_N}(q_1,...,q_N)$ are such that

$$\left|\Psi_{\sigma_1...\sigma_N}(q_1,...,q_N)\right|^2 = (2\epsilon_0)^N C^{(N)}_{\sigma_1...\sigma_N}(q_1,...,q_N). \tag{4}$$

This relation shows that in a N-photon system, all correlations in space, time and polarisation that one *could* measure in a photon counting experiment are "encoded" in the complex field $\Psi_{\sigma_1...\sigma_N}(q_1,...,q_N)$. This includes quantum effects resulting from particle entanglement and quantum interferences. Therefore, when applying the WFF, the only challenge consists in finding out how the multi-particle wave function propagates in a given setup from sources to detectors.

Generalised Huygens-Fresnel principle

In order to understand how the N-photon correlations spread in space and time, we first consider a situation in which the wave function propagates in free-space. We also make the simplifying assumption that we deal with paraxial states of light, in which case polarisation does not change much during propagation. Therefore we restrict the following discussion to a single polarisation component (dropping all the polarisation-related indexes). Considering photons propagating along the z-axis, we find that the wave function at different space-time points can be calculated from its past values in some fixed "preparation plane" $z=\zeta$ using the following integral relation

$$\begin{aligned}\Psi(\mathbf{r}_1,t_1,\ldots,\mathbf{r}_N,t_N) &= \frac{1}{(2\pi c)^N}\iint \mathrm{d}^2\rho_1^\perp \ldots \iint \mathrm{d}^2\rho_N^\perp \\ &\times \frac{\frac{\mathrm{d}}{\mathrm{d}t_1}\cdots\frac{\mathrm{d}}{\mathrm{d}t_N}\Psi(\boldsymbol{\rho}_1,t_1-\frac{|\mathbf{r}_1-\boldsymbol{\rho}_1|}{c},\ldots,\boldsymbol{\rho}_N,t_N-\frac{|\mathbf{r}_N-\boldsymbol{\rho}_N|}{c})}{|\mathbf{r}_1-\boldsymbol{\rho}_1|\cdots|\mathbf{r}_N-\boldsymbol{\rho}_N|}.\end{aligned} \tag{5}$$

We call it the generalised Huygens-Fresnel principle (HFP) for N-photon wave functions since it reduces to the standard HFP for $N=1$ [8]. In Eq. (5), $\boldsymbol{\rho}_j=(\xi_j,\eta_j,\zeta)$ ($j\in\{1,...,N\}$) are vectors representing points in the ζ-plane and $\boldsymbol{\rho}_j^\perp=(\xi_j,\eta_j)$ are their transverse components. In the optical domain, photons can be usually considered as quasi-monochromatic. Therefore, the wave function can be written

$$\Psi(\mathbf{r}_1,t_1,\ldots,\mathbf{r}_N,t_N) = a(\mathbf{r}_1,t_1,\ldots,\mathbf{r}_N,t_N)\mathrm{e}^{-\mathrm{i}2\pi c(\frac{t_1}{\lambda_1}+\ldots+\frac{t_N}{\lambda_N})}, \tag{6}$$

where $a(\mathbf{r}_1,t_1,\ldots,\mathbf{r}_N,t_N)$ is a slowly varying function of time and λ_j ($j\in\{1,...,N\}$) are the central wavelengths of the photons. Note that nothing here prevents some of the photons from having the same central wavelength nor even being indistinguishable. Inserting (6) in Eq. (5) and taking into account that $a(\mathbf{r}_1,t_1,\ldots,\mathbf{r}_N,t_N)$ is slowly varying with time, one obtains the quasi-monochromatic form of the HFP:

$$\begin{aligned}a(\mathbf{r}_1,t_1,\ldots,\mathbf{r}_N,t_N) &= \frac{(-\mathrm{i})^N}{\lambda_1\ldots\lambda_N}\iint \mathrm{d}^2\rho_1^\perp \ldots \iint \mathrm{d}^2\rho_N^\perp \\ a(\boldsymbol{\rho}_1,t_1-\frac{|\mathbf{r}_1-\boldsymbol{\rho}_1|}{c},\ldots,\boldsymbol{\rho}_N,t_N-\frac{|\mathbf{r}_N-\boldsymbol{\rho}_N|}{c})&\ \frac{\mathrm{e}^{\mathrm{i}\frac{2\pi}{\lambda_1}|\mathbf{r}_1-\boldsymbol{\rho}_1|}}{|\mathbf{r}_1-\boldsymbol{\rho}_1|}\cdots\frac{\mathrm{e}^{\mathrm{i}\frac{2\pi}{\lambda_N}|\mathbf{r}_N-\boldsymbol{\rho}_N|}}{|\mathbf{r}_N-\boldsymbol{\rho}_N|}.\end{aligned} \tag{7}$$

Propagation of entanglement in parametric down-conversion

We now show how to use Eq. (7) in order to calculate the propagation of the entanglement of a photon pair generated by down-conversion in a thin non linear crystal. Such photons are created simultaneously and at the same transverse position, therefore $a(\boldsymbol{\rho}_1, t_1, \boldsymbol{\rho}_2, t_2) \propto \delta(t_1 - t_2)\delta(\boldsymbol{\rho}_1^{\perp} - \boldsymbol{\rho}_2^{\perp})$ in the output plane of crystal plane. Choosing the crystal plane as the Oxy plane, free space propagation behind the crystal results in the following two-photon amplitude:

$$a(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2) \propto \frac{-1}{\lambda_1 \lambda_2} \iint \mathrm{d}^2 \rho^{\perp} \delta(t_1 - \frac{|\mathbf{r}_1 - \boldsymbol{\rho}^{\perp}|}{c} - t_2 + \frac{|\mathbf{r}_2 - \boldsymbol{\rho}^{\perp}|}{c}) \frac{\mathrm{e}^{\mathrm{i}\frac{2\pi}{\lambda_1}|\mathbf{r}_1 - \boldsymbol{\rho}^{\perp}|}}{|\mathbf{r}_1 - \boldsymbol{\rho}^{\perp}|} \frac{\mathrm{e}^{\mathrm{i}\frac{2\pi}{\lambda_2}|\mathbf{r}_2 - \boldsymbol{\rho}^{\perp}|}}{|\mathbf{r}_2 - \boldsymbol{\rho}^{\perp}|}. \quad (8)$$

This formula shows that the detection of the λ_1-photon at point $\mathbf{r}_1$ at time t_1 automatically gives the λ_2-photon a spherical wavefront and a duration that depends on the crystal size. If the λ_2-photon is detected first a similar effect will be observed with the λ_1-photon. By placing lenses in the system, spherical wavefronts can be focused leading to the well-known quantum-imaging effect demonstrated by Pittman in [9]. Our generalised HFP accounts for this quantum effect in a much simpler way than any previous theoretical description.

Conclusion

We showed that, with a slight modification, the Bialynicki-Birula photon wave function can be used to calculate space, time and polarisation correlations that arise in a N-photon system. We also established a generalised Huygens-Fresnel principal that allows us to compute the propagation of the wave function in the paraxial approximation. We illustrate the power of the new formalism by calculating the propagation of entanglement in a two-photon system generated by parametric down-conversion.

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